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On improving the effectiveness of periodic solutions of the NLS and DNLS equations

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Abstract. A method for improving the effectiveness of one-phase periodic solutions of integrable equations is suggested, which is based on the explicit determination of the locus of the auxiliary spectrum point. The method is applied to the nonlinear Schrödinger equation and the derivative nonlinear Schrödinger equation. It is shown that the relations between polynomial $P(\lambda)$, defining the solution in the inverse transform method, and its algebraic resolvents play an important role in the theory.

1. Introduction

The nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t + u_{xx} + 2|u|^2 u = 0 \tag{1.1}$$

and derivative nonlinear Schrödinger (DNLS) equation

$$\mathrm{i}u_t + u_{xx} \pm 2\mathrm{i}(|u|^2 u)_x = 0 \tag{1.2}$$

are very important both in the mathematical theory of integrable equations and in various physical applications (Newell 1985). Discovery of the integrability of the NLS equation by the inverse scattering transform (IST) method and finding its soliton solutions (Zakharov and Shabat 1971) promoted to a large extent the understanding of the generality of this method and its subsequent fast development. Integrability of the DNSL equation was established by Kaup and Newell (1978) with derivation of the corresponding soliton solutions.

For physical applications it is necessary to know not only soliton solutions, but also the periodic solutions of these equations. The general principles of obtaining the periodic solutions of integrable equations were first developed for the KdV equation (Novikov 1974, Dubrovin 1975, Its and Matveev 1975, McKean and van Moerbeke 1975). Then this method was used by Its and Kotlyarov (1976) for the NLS equation. But the formulae obtained have proved to be less effective than for the KdV equation even in the simplest case of one-phase periodic solutions. The reasons for this phenomenon can be summarised as follows.

As is well known (Newell 1985), in the IST method the solutions of the integrable equation are determined by the spectrum of the corresponding linear operator L. For example, in the case of the KdV equation the L operator coincides with the quantum mechanical Schrödinger operator, the spectrum of which in the periodic case consists of some number of stability bands separated by lacunae. The corresponding Bloch function considered as a function of spectral parameter λ is single-valued on the two-sheet Riemann surface, obtained by joining together two complex planes of spectral parameter λ with cuts along the lacunae. The spectrum does not depend on time, and evolution of the solutions is determined by the movement of the points of the auxiliary spectrum, where the Bloch function has zeros, along this Riemann surface. It is easy to show in the KdV case that each point of the auxiliary spectrum lies inside 'its' lacuna, so they can move along cycles around lacunae only. The same picture holds for all integrable equations with self-adjoint corresponding operator L, e.g. for the NLS equation with a minus sign before the last term in (1.1). However, for the NLS equation (1.1), and the DNLS equation (1.2) and for other physically important equations the operator L is not self-adjoint, hence the spectrum may be complex and the term 'lacuna' no longer has a clear meaning. Therefore the loci of the auxiliary spectrum points are not prescribed beforehand, and their determination is not a simple problem (Ma and Ablowitz 1981, Previato 1985). In this paper we shall solve this problem in the case of NLS and DNLS one-phase periodic solutions. This complements the method of Its and Kotlyarov (1976) and will permit one to obtain the solutions under consideration in a form convenient for applications.

2. The NLS equation

2.1. Periodic solutions

The comprehensive exposition of the IST method for the NLS equation was given by Tracy and Chen (1988). We will give here some of the main relations relevant to the one-phase case, in which the NLS periodic solution is determined by the elliptic Riemann surface

$$l^2 = P(\lambda) \qquad P(\lambda) = \prod_{i=1}^{4} (\lambda - \lambda_i) = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4 \tag{2.1}$$

where s_i are the usual symmetrical functions of the zeros λ_i of polynomial $P(\lambda)$. As is known, these zeros consist of two complex conjugate pairs. There is only one point of the auxiliary spectrum $\mu(x, t)$ whose movement along its locus is determined by the equations

$$\frac{\partial\mu}{\partial x} = \pm 2\mathrm{i}\sqrt{P(\mu)} \qquad \frac{\partial\mu}{\partial t} = s_1 \frac{\partial\mu}{\partial x} \tag{2.2}$$

where signs $\pm$ correspond to the two sheets of the Riemann surface. It is evident that μ depends only on the phase

$$W = x + s_1 t + W_0. \tag{2.3}$$

If $\mu(x, t)$ is already known, then the solution $u(x, t)$ of the NLS equation can be obtained from the equations

$$\frac{\partial \ln u}{\partial x} = 2\mathrm{i}(\mu - \tfrac{1}{2}s_1) \qquad \frac{\partial \ln u}{\partial t} = 2\mathrm{i}(s_2 - \tfrac{3}{4}s_1^2) + 2\mathrm{i}s_1\mu. \tag{2.4}$$

As has been emphasised by Its and Kotlyarov (1976) (see also Tracy *et al* 1984), the initial conditions for the solutions of equations (2.2) and (2.4) must satisfy the constraint

$$P(\lambda) = f^2(\lambda) + |u|^2(\lambda - \mu)(\lambda - \mu^*) \tag{2.5}$$

where $|u|^2 = |u(x,t)|^2$, $f(\lambda) = \lambda^2 - f_1\lambda + f_2$. The usual method of approach (Its and Kotlyarov 1976, Tracy and Chen 1988) involves an integration of equations (2.2) and (2.4) with the initial conditions satisfying the constraint (2.5). We shall show that for one-phase solutions equation (2.5) can be resolved explicitly, which leads to more effective expressions.

The coefficients of polynomial $f(\lambda)$ and the values of μ, μ^* are, obviously, the functions of $\nu = |u|^2$. During the evolution of $u(x,t)$ according to the NLS equation, the magnitude of $|u(x,t)|^2$ varies, so let us choose to take the initial value with the corresponding values of μ, μ^*. Hence (2.5) gives us, in fact, the locus of μ, μ^*, and $\nu = |u(x,t)|^2$ is the natural coordinate along this locus. The trajectory of μ must be closed and ν must oscillate between two positive values. Both the locus (i.e. trajectory of μ) and the period of oscillations of ν are determined by the distribution of the zeros of the polynomial $P(\lambda)$, and in the one-phase case they can be easily found as follows.

Comparing the coefficients of the λ^k on both sides of (2.5), we obtain the equations

$$\begin{aligned} &s_1 = 2f_1 \qquad s_2 = f_1^2 + 2f_2 + \nu \\ &s_3 = 2f_1f_2 + \nu(\mu+\mu^*) \qquad s_4 = f_2^2 + \nu\mu\mu^*. \end{aligned} \tag{2.6}$$

These equations give

$$f_1 = \tfrac{1}{2}s \qquad f_2 = -\tfrac{1}{16}s^2 + \tfrac{1}{2}(p-\nu) \tag{2.7}$$

$$\mu+\mu^* = \tfrac{1}{2}s - (q/\nu) \qquad \mu\mu^* = -(1/4\nu)[\nu^2 - 2\nu(p+\tfrac{1}{8}s^2) + sq + p^2 - 4r] \tag{2.8}$$

where

$$\begin{aligned} &s = s_1 \qquad p = s_2 - \tfrac{3}{8}s_1^2 \qquad q = \tfrac{1}{2}s_1(s_2 - \tfrac{1}{4}s_1^2) - s_3 \\ &r = s_4 + \tfrac{1}{16}s_1^2(s_2 - \tfrac{3}{16}s_1^2) - \tfrac{1}{4}s_1s_3. \end{aligned} \tag{2.9}$$

Thus we see that μ and μ^* are the solutions of the quadratic equation whose coefficients are given by the expressions (2.8). It is remarkable that its discriminant is equal to $R(\nu)/\nu^2$, where

$$R(\nu) = \nu^3 - 2p\nu^2 + (p^2 - 4r)\nu + q^2 \tag{2.10}$$

is the cubic resolvent of the polynomial $P(\lambda)$ (van der Waerden 1971). Thus

$$\mu, \mu^* = \tfrac{1}{4}s - \frac{q \pm \sqrt{R(\nu)}}{2\nu}$$

and we have proved the identity

$$\begin{aligned} &\lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4 \\ &\quad = [(\lambda - \tfrac{1}{4}s)^2 + \tfrac{1}{2}(p-\nu)]^2 + \nu\left(\lambda - \frac{s}{4} + \frac{q - \sqrt{R(\nu)}}{2\nu}\right)\left(\lambda - \frac{s}{4} + \frac{q + \sqrt{R(\nu)}}{2\nu}\right). \end{aligned} \tag{2.11}$$

As is known (van der Waerden 1971), the zeros ν_1, ν_2, ν_3 of the cubic resolvent $R(\nu)$ are related to the zeros λ_i, $i = 1, 2, 3, 4$, of the polynomial $P(\lambda)$ by the formulae

$$\begin{aligned} &\nu_1 = -\tfrac{1}{4}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)^2 \qquad \nu_2 = -\tfrac{1}{4}(\lambda_1 - \lambda_2 - \lambda_2 + \lambda_4)^2 \\ &\nu_3 = -\tfrac{1}{4}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2. \end{aligned} \tag{2.12}$$

In the case of the periodic solutions of the NLS equations when $\lambda_1, \ldots, \lambda_4$ consist of the two complex conjugate pairs

$$\lambda_1 = \alpha + \mathrm{i}\gamma \qquad \lambda_2 = \beta + \mathrm{i}\delta \qquad \lambda_3 = \alpha - \mathrm{i}\gamma \qquad \lambda_4 = \beta - \mathrm{i}\delta \tag{2.13}$$

the formulae (2.12) yield

$$\nu_1 = -(\alpha - \beta)^2 \qquad \nu_2 = (\gamma - \delta)^2 \qquad \nu_3 = (\gamma + \delta)^2. \tag{2.14}$$

This means that ν oscillates in the interval $\nu_2 \leqslant \nu \leqslant \nu_3$, where the resolvent $R(\nu)$ is negative, and for the trajectory of μ (its locus) we have

$$\mu = \frac{s}{4} - \frac{q + \mathrm{i}\sqrt{-R(\nu)}}{2\nu} \tag{2.15}$$

and μ^* is the complex conjugate of this expression. If we wish to separate the real and imaginary parts of μ ($\mu = \mu' + \mathrm{i}\mu''$), then it is easy to find that μ' and μ'' are related by the cubic curve equation. The graph of this curve is shown in figure 1. Its branch in the form of a closed oval corresponds to the periodic solution. This oval lies on two sheets of the Riemann surface (2.1) going from one sheet to another through the cuts which connects λ_1 with λ_2 and λ_3 with λ_4, respectively.

After finding the locus of $\mu(x, t)$, we may proceed to the solution of equation (2.2). During the movement of $\mu(x, t)$ along curve (2.15) the last term in the identity (2.11) vanishes, so that (2.2) and (2.3) provide

$$\frac{\mathrm{d}\mu}{\mathrm{d}(2W)} = -\mathrm{i}\sqrt{P(\mu)} = -\mathrm{i}f(\mu). \tag{2.16}$$

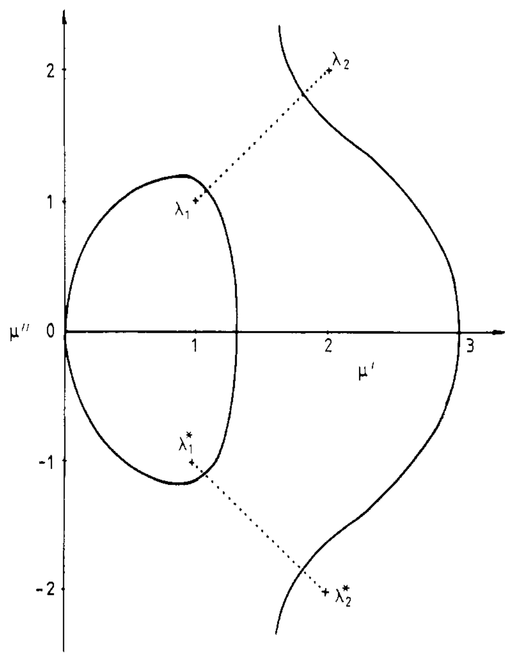

Figure 1. A plot of the curve (2.15) in the complex μ plane for $\lambda_1 = 1 + \mathrm{i}$, $\lambda_2 = 2 + 2\mathrm{i}$.

Since the variables μ and ν are related by (2.15), we shall first search for ν instead of μ. Differentiation of $P(\mu)=f^2(\mu)$ with respect to ν gives

$$\frac{\mathrm{d}P}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\nu}=2f(\mu)\left[2\left(\mu-\frac{s}{4}\right)\frac{\mathrm{d}\mu}{\mathrm{d}\nu}-\frac{1}{2}\right].$$

From this we find the derivative

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}=-\frac{2\nu}{f(\mu)}\left(\mu-\frac{s}{4}+\frac{q}{2\nu}\right)=-\frac{2\nu}{f(\mu)}\mathrm{i}\mu''=\frac{\mathrm{i}}{f(\mu)}\sqrt{-R(\nu)}. \tag{2.17}$$

Multiplying (2.16) by (2.17), we obtain the equation for ν:

$$\frac{\mathrm{d}\nu}{\mathrm{d}(2W)}=\sqrt{-R(\nu)}. \tag{2.18}$$

Since $R(\nu)$ is the cubic polynomial, the solution of (2.18) can be expressed through the Weierstrass $\wp$ function. To this end let us express a resolvent in Weierstrass form

$$R(\nu)=-16(4\xi^3-g_2\xi-g_3)=-16\times 4(\xi-e_1)(\xi-e_2)(\xi-e_3) \tag{2.19}$$

where

$$\xi=\tfrac{1}{6}p-\tfrac{1}{4}\nu \qquad \nu=\tfrac{2}{3}p-4\xi \tag{2.20}$$

$$e_1=\tfrac{1}{12}(\nu_2+\nu_3-2\nu_1) \qquad e_2=\tfrac{1}{12}(\nu_1+\nu_3-2\nu_2) \qquad e_3=\tfrac{1}{12}(\nu_1+\nu_2-2\nu_3) \tag{2.21}$$

and the invariants

$$g_2=r+\tfrac{1}{12}p^2 \qquad -g_3=\tfrac{1}{6}p[r-(\tfrac{1}{6}p)^2]-\tfrac{1}{16}q^2$$

coincide with those of the polynomial $P(\lambda)$. Substituting (2.19) and (2.20) into (2.18), we find

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}(2W)}\right)^2=4\xi^3-g_2\xi-g_3$$

which gives

$$\xi=\wp(2W+c) \qquad \nu=\tfrac{2}{3}p-4\wp(2W+c). \tag{2.22}$$

An integration constant c is determined by the initial conditions, which we shall choose as follows: $\nu=\nu_3$ at $W=0$, i.e. $\wp(c)=e_3$, and hence $c=\omega'$ (ω and ω' are half-periods of the $\wp$ function). Another choice of initial conditions corresponds to adding a constant term to the phase (2.3). In the one-phase case a constant additive in phase has no physical sense, and we shall take $W_0=0$ for simplification of the formulae. Thus, the μ trajectory is now expressed through the phase W which depends linearly on time t and coordinate x:

$$\mu(x,t)=\frac{s}{4}+\frac{q}{8}\frac{1}{\wp(2W+\omega')-\frac{1}{6}p}-\frac{\mathrm{i}}{2}\frac{1}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}(2W)}. \tag{2.23}$$

Now we shall turn to calculation of $u(x,t)$. From (2.4) we have

$$u(x,t)=\exp[2\mathrm{i}(s_2-\tfrac{1}{4}s_1^2)t]\hat{u}(W) \tag{2.24}$$

provided

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}W}=2\mathrm{i}(\mu-\tfrac{1}{2}s_1)\hat{u}.$$

Substitution of (2.23) into the last equation leads to

$$\frac{\mathrm{d}\log\hat{u}}{\mathrm{d}(2W)}=\frac{1}{2}\frac{\mathrm{d}\log\nu}{\mathrm{d}(2W)}-\frac{\mathrm{i}s}{2}+\frac{\mathrm{i}q}{8}\frac{1}{\wp(2W+\omega')-\frac{1}{6}p}. \tag{2.25}$$

This equation can be easily integrated by means of the formula (Gradshtein and Ryzhik 1980)

$$\int\frac{\mathrm{d}z}{\wp(z)-\wp(\varkappa)}=\frac{1}{\wp'(\varkappa)}\left(2z\zeta(\varkappa)+\ln\frac{\sigma(z-\varkappa)}{\sigma(z+\varkappa)}\right) \tag{2.26}$$

where ζ and σ are the Weierstrass functions. Choosing $\varkappa$ according to

$$\wp(\varkappa)=\tfrac{1}{6}p \tag{2.27}$$

we find $\wp'(\varkappa)$ from the differential equation for $\wp(\varkappa)$:

$$\wp'(\varkappa)=-\tfrac{1}{4}\mathrm{i}q. \tag{2.28}$$

After a simple calculation with the use of identity (Erdelyi *et al* 1955)

$$\wp(2W+\omega')-\wp(\varkappa)=-\frac{\sigma(2W+\omega'+\varkappa)\sigma(2W+\omega'-\varkappa)}{\sigma^2(2W+\omega')\sigma^2(\varkappa)} \tag{2.29}$$

we obtain the periodic solution for the NLS equation in the form

$$u(x,t)=2\exp[-\mathrm{i}(\alpha+\beta)x-2\mathrm{i}(\alpha^2+\beta^2-\gamma^2-\delta^2)t-2\zeta(\varkappa)W-\eta'\varkappa]\times\frac{\sigma(2W+\varkappa+\omega')}{\sigma(2W+\omega')\sigma(\varkappa)} \tag{2.30}$$

where $\eta=\zeta(\omega)$, $\eta'=\zeta(\omega')$. Let us introduce a parameter χ such that

$$\varkappa=\omega+\chi. \tag{2.31}$$

Then a simple calculation gives

$$\zeta(\varkappa)=\zeta(\chi)+\eta-\tfrac{1}{2}(\nu_2\nu_3/\nu_1)^{1/2}.$$

From the σ function in (2.30) we pass to the ϑ functions by means of the relation

$$\sigma(z)=2\omega\exp\left(\frac{\eta z^2}{2\omega}\right)\frac{\vartheta_1(v)}{\vartheta_1'(0)}\qquad v=\frac{z}{2\omega}$$

which gives

$$\begin{aligned}u(x,t)=(\gamma+\delta)\exp\bigg[&-\mathrm{i}(\alpha+\beta)x-2\mathrm{i}(\alpha^2+\beta^2-\gamma^2-\delta^2)t\\&-2W\left(\zeta(\chi)-\frac{\eta\chi}{\omega}+\frac{\mathrm{i}}{2}\frac{\gamma^2-\delta^2}{|\alpha-\beta|}\right)\bigg]\frac{\vartheta_4(0)\vartheta_3[(2W+\chi)/2\omega]}{\vartheta_4(W/\omega)\vartheta_3(\chi/2\omega)}\\W&=x+2(\alpha+\beta)t\end{aligned} \tag{2.32}$$

where it is supposed without loss of generality that $\gamma\geqslant\delta$. From (2.27) and (2.31) we find

$$\wp(\chi)-e_3=\frac{e_1-e_3}{\mathrm{sn}^2(\sqrt{e_1-e_3}\,\chi,k)}=\frac{\nu_2}{\nu_1}(e_1-e_3)$$

so that

$$\mathrm{sn}(\sqrt{e_1-e_3}\,\chi,k)=(\nu_1/\nu_2)^{1/2} \tag{2.33}$$

where

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{\nu_3 - \nu_2}{\nu_3 - \nu_1} = \frac{4\gamma\delta}{(\gamma+\delta)^2 + (\alpha-\beta)^2}. \tag{2.34}$$

Expression (2.33) gives

$$[(\gamma+\delta)^2 + (\alpha-\beta)^2]^{1/2} \frac{\chi}{2} = \mathrm{i} F(\varphi, k') = \mathrm{i} \int_0^{\sin\varphi} \frac{\mathrm{d}z}{[(1-z^2)(1-k'^2 z^2)]^{1/2}} \tag{2.35}$$

where $k'^2 = 1 - k^2$ and the angle φ has a very simple geometrical sense

$$\sin\varphi = \frac{|\alpha-\beta|}{[(\alpha-\beta)^2 + (\gamma-\delta)^2]^{1/2}}. \tag{2.36}$$

It is easy to obtain the simple expression for the variation of the absolute value of the field $u(x, t)$ from the second formula (2.22):

$$\begin{aligned} |u(x,t)|^2 &= \nu_3 + (\nu_2 - \nu_3)\,\mathrm{sn}^2(\sqrt{\nu_3 - \nu_2}\, W, k) \\ &= (\gamma+\delta)^2 - 4\gamma\delta\, \mathrm{sn}^2\{[(\gamma+\delta)^2 + (\alpha-\beta)^2]^{1/2} W, k\}. \end{aligned} \tag{2.37}$$

The formulae (2.32)-(2.37) express the general periodic solution of the NLS equation (2.1) in a form which is relatively simple and suitable for use.

2.2. *Limiting cases*

Let us discuss some important limiting cases of the general solution.

Let $\beta \to \alpha$, i.e. all zeros lie on the one vertical line. In this case (2.35) gives

$$\chi \simeq 2\mathrm{i} \frac{|\alpha-\beta|}{\gamma^2 - \delta^2} \to 0$$

so that

$$\zeta(\chi) \simeq \frac{1}{\chi} \simeq -\frac{\mathrm{i}}{2} \frac{\gamma^2 - \delta^2}{|\alpha-\beta|}$$

and the term in large parentheses in (2.32) vanishes. The ratio of the ϑ functions is equal in this limit to the Jacobi dn function, hence we obtain

$$u(x,t) = (\gamma+\delta) \exp[-2\mathrm{i}\alpha x - 2\mathrm{i}(2\alpha^2 - \gamma^2 - \delta^2)t]\, \mathrm{dn}\left((\gamma+\delta)W, \frac{2\sqrt{\gamma\delta}}{\gamma+\delta}\right) \tag{2.38}$$

where $W = x + 4\alpha t$. It is convenient to go to another modulus of the elliptic functions by means of the identity

$$(1+k)\,\mathrm{dn}\left((1+k)z, \frac{2\sqrt{k}}{1+k}\right) = \mathrm{dn}(2z, k) + k\,\mathrm{cn}(2z, k).$$

Then (2.38) takes the form

$$u(x,t) = \exp[-2\mathrm{i}\alpha x - 2\mathrm{i}(2\alpha^2 - \gamma^2 - \delta^2)t][\gamma\,\mathrm{dn}(2\gamma W, \delta/\gamma) + \delta\,\mathrm{cn}(2\gamma W, \delta/\gamma)]. \tag{2.39}$$

At $\delta \to \gamma$, when two pairs of zeros λ_i coalesce into one pair, we obtain the well known soliton solution

$$u(x,t) = 2\gamma \exp[-2\mathrm{i}\alpha x - 4\mathrm{i}(\alpha^2 - \gamma^2)t]\, \mathrm{sech}[2\gamma(x + 4\alpha t)]. \tag{2.40}$$

Now let us discuss another degenerate case where $\alpha \neq \beta$ but $\delta \to \gamma$, i.e. the zeros λ_i lie on two horizontal lines. According to (2.36), we have $\sin \varphi = 1$, so that (2.35) gives $\chi = \omega'$, $\zeta(\chi) = \eta'$ and simple calculations lead to

$$u(x, t) = 2\gamma \exp[-\mathrm{i}(\alpha+\beta)x - 2\mathrm{i}(\alpha^2+\beta^2-2\gamma^2)t] \times \mathrm{cn}\left([4\gamma^2+(\alpha-\beta)^2]^{1/2} W, \frac{2\gamma}{[4\gamma^2+(\alpha-\beta)^2]^{1/2}}\right) \tag{2.41}$$

where $W = x + 2(\alpha+\beta)t$. In the limit $\beta \to \alpha$ this periodic solution goes to the same soliton solution (2.40).

As another example, let us discuss the case of small wave modulations with $\delta \ll \gamma$. Now we have a small parameter

$$k^2 \simeq \frac{4\gamma\delta}{\gamma^2+(\alpha-\beta)^2} \propto \frac{\delta}{\gamma} \ll 1 \tag{2.42}$$

and (2.32) should be expanded with respect to its degrees. From (2.35) we obtain (for $k' \to 1$)

$$\chi \simeq \frac{\mathrm{i}}{\sqrt{\gamma^2+(\alpha-\beta)^2}} \log \frac{1+\sin\varphi_0}{1-\sin\varphi_0} \tag{2.43}$$

where $\sin\varphi_0$ is determined by (2.36) with $\delta = 0$. At $k \to 0$ we have

$$\zeta(\chi) \simeq \tfrac{1}{2}e_1\chi + (\tfrac{3}{2}e_1)^{1/2} \cot[(\tfrac{3}{2}e_1)^{1/2}\chi] \qquad e_1 = [\gamma^2+(\alpha-\beta)^2]^{1/2} \tag{2.44}$$

and η/ω can be calculated by means of the formula

$$\frac{\eta}{\omega} = (e_1 - e_3)\frac{E(k)}{K(k)} - e_1$$

which at $k \to 0$ gives $\eta/\omega \simeq \frac{1}{2}e_1$. Substitution of (2.43) into (2.44) gives for the brackets in (2.32) the expression

$$\zeta(\chi) - \frac{\eta\chi}{\omega} + \frac{\mathrm{i}}{2}\frac{\gamma^2-\delta^2}{|\alpha-\beta|} \simeq -\frac{\mathrm{i}}{2}|\alpha-\beta|.$$

We must substitute (2.43) and

$$\omega = \frac{K(k)}{\sqrt{e_1-e_3}} \simeq \frac{\pi}{\sqrt{\gamma^2+(\alpha-\beta)^2}}$$

into the arguments of the ϑ functions and expand them in powers of k^2. After simple calculations we find

$$u(x, t) = \exp[-\mathrm{i}(\alpha+\beta)x - 2\mathrm{i}(\alpha^2+\beta^2-\gamma^2)t + \mathrm{i}|\alpha-\beta|W] \times\left[\gamma + \delta\left(\cos(2\sqrt{\gamma^2+(\alpha-\beta)^2}\,W) - \frac{\mathrm{i}|\alpha-\beta|}{\sqrt{\gamma^2+(\alpha-\beta)^2}} \times \sin(2\sqrt{\gamma^2+(\alpha-\beta)^2}\,W)\right)\right] \tag{2.45}$$

where $W = x + 2(\alpha+\beta)t$. This expression gives the solution of the NLS equation with small modulation and coincides with the results of perturbation theory.

We see that the general expression (2.32) contains all known results as particular limiting cases, and relates them to the distribution of spectral data λ_i, $i=1,\ldots,4$, on the complex plane.

The calculations of this section use the relation between polynomial $P(\lambda)$ and its resolvent $R(\nu)$. It is natural to ask how this method should be modified for other integrable equations. We shall answer this question for the case of the DNLS equation in the next section.

3. DNLS equation

We shall discuss the DNLS equation (1.2) with a minus sign before the last term

$$\mathrm{i}u_t + u_{xx} - 2\mathrm{i}(|u|^2 u)_x = 0. \tag{3.1}$$

This sign can be easily inverted by means of simple substitutions.

3.1. *The general equations of* IST

Integrability of (3.1) is based on the possibility of representing this equation as a compatibility condition of two systems of linear equations containing an arbitrary spectral parameter λ. We shall take these systems in the form (Wadati *et al* 1979)

$$\frac{\partial\psi_1}{\partial x} = -2\lambda^2\psi_1 + 2\lambda u(x,t)\psi_2 \qquad \frac{\partial\psi_2}{\partial x} = 2\lambda u^*(x,t)\psi_1 + 2\lambda^2\psi_2 \tag{3.2}$$

and

$$\begin{aligned}\frac{\partial\psi_1}{\partial t} &= -[8\mathrm{i}\lambda^4 + 4\mathrm{i}|u|^2\lambda^2]\psi_1 + [8\lambda^3 u + (2\mathrm{i}u_x + 4|u|^2 u)\lambda]\psi_2 \\ \frac{\partial\psi_2}{\partial t} &= [8\lambda^3 u^* + (-2\mathrm{i}u_x^* + 4|u|^2 u^*)\lambda]\psi_1 + [8\mathrm{i}\lambda^4 + 4\mathrm{i}|u|^2\lambda^2]\psi_2.\end{aligned} \tag{3.3}$$

These linear systems have two basic solutions $\psi=(\psi_1,\psi_2)$ and $\varphi=(\varphi_1,\varphi_2)$, which satisfy the different boundary conditions. Now, it is convenient to pass to linear systems for 'squared basic functions'

$$f = -\tfrac{1}{2}\mathrm{i}(\varphi_1\psi_2 + \varphi_2\psi_1) \qquad g = \varphi_1\psi_1 \qquad h = -\varphi_2\psi_2. \tag{3.4}$$

These systems have the form

$$\frac{\partial f}{\partial x} = -2\mathrm{i}\lambda u^* g + 2\mathrm{i}\lambda u h \qquad \frac{\partial g}{\partial x} = 4\mathrm{i}\lambda u f - 4\mathrm{i}\lambda^2 g \qquad \frac{\partial h}{\partial x} = -4\mathrm{i}\lambda u^* f + 4\mathrm{i}\lambda^2 h \tag{3.5}$$

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\mathrm{i}[8\lambda^3 u^* + (-2\mathrm{i}u_x^* + 4|u|^2 u^*)\lambda]g + \mathrm{i}[8\lambda^3 u + (2\mathrm{i}u_x + 4|u|^2 u)\lambda]h \\ \frac{\partial g}{\partial t} &= 2\mathrm{i}[8\lambda^3 u + (2\mathrm{i}u_x + 4|u|^2 u)\lambda]f - 8\mathrm{i}(2\lambda^4 + |u|^2\lambda^2)g \\ \frac{\partial h}{\partial t} &= -2\mathrm{i}[8\lambda^3 u^* + (-2\mathrm{i}u_x^* + 4|u|^2 u^*)\lambda]f + 8\mathrm{i}(2\lambda^4 + |u|^2\lambda^2)h.\end{aligned} \tag{3.6}$$

It is easy to check that the expression

$$f^2 - gh = P \tag{3.7}$$

does not depend on x and t, so that P is a function of λ only. Periodic solutions are distinguished by the condition that $P=P(\lambda)$ be a polynomial in λ. Seeking the solutions of systems (3.5) and (3.6) in the form of polynomials in λ, it is easy to find that $P(\lambda)$ can contain only the even degrees of λ. Non-trivial solutions exist if the degree of $P(\lambda)$ is equal to or is more than 6. The degrees 6 and 8 correspond to the one-phase periodic solutions in which we are interested. It will be clear from the following that solutions corresponding to the sixth degree of $P(\lambda)$ are particular cases of solutions corresponding to the eighth degree of $P(\lambda)$. Therefore we assume that $P(\lambda)$ has the form

$$P(\lambda)=\prod_{i=1}^{4}(\lambda^2-\lambda_i^2)=\lambda^8-s_1\lambda^6+s_2\lambda^4-s_3\lambda^2+s_4 \tag{3.8}$$

where $\pm\lambda_i$ are the zeros of the polynomial. Then (3.5) and (3.6) lead to the expressions

$$f=\lambda^4-f_1\lambda^2+f_2 \qquad g=u\lambda(\lambda^2-\mu) \qquad h=u^*\lambda(\lambda^2-\mu^*) \tag{3.9}$$

and

$$\frac{\partial u}{\partial x}=-4\mathrm{i}u(f_1-\mu) \qquad \frac{\partial u}{\partial t}=8\mathrm{i}u[2f_2-(f_1-\mu)(2f_1+|u|^2)] \tag{3.10}$$

where the quantities $f_1, f_2, |u|^2, \mu, \mu^*$ are connected by the following constraint, which is a result of (3.7):

$$(\lambda^4-f_1\lambda^2+f_2)^2-|u|^2\lambda^2(\lambda^2-\mu)(\lambda^2-\mu^*)=P(\lambda). \tag{3.11}$$

The variable μ is called the auxiliary spectrum point of the eigenvalue problem (3.5) with periodic boundary conditions. The dependence of μ on x and t can be obtained from (3.5) and (3.6) if one puts $\lambda^2=\mu$ and takes into account that $f(\mu^{1/2})=\sqrt{P(\mu^{1/2})}$:

$$\frac{\partial\mu}{\partial x}=\pm4\mathrm{i}\sqrt{P(\mu^{1/2})} \qquad \frac{\partial\mu}{\partial t}=\pm8\mathrm{i}(2f_1+|u|^2)\sqrt{P(\mu^{1/2})}. \tag{3.12}$$

We see that μ moves over an elliptic Riemann surface (l,λ) defined by the equation $l^2=P(\lambda^{1/2})$.

3.2. *Periodic solutions of the* DNLS *equation*

As in the case of the NLS equation, we see that the constraint (3.11) can be considered as an equation for the locus of μ in the complex λ plane, and the variable $\nu=|u|^2$ is the natural coordinate along this locus. Comparing the coefficients of λ^k on both sides of (3.11), we have

$$\begin{aligned} s_1&=2f_1+\nu \qquad & s_2&=f_1^2+2f_2+\nu(\mu+\mu^*)\\ s_3&=2f_1f_2+\nu\mu\mu^* \qquad & s_4&=f_2^2. \end{aligned} \tag{3.13}$$

One can obtain from these equations the expression for μ

$$\mu=\frac{1}{8\nu}[4s_2\pm8\sqrt{s_4}-(\nu-s_1)^2+\mathrm{i}\sqrt{-R(\nu)}] \tag{3.14}$$

where $R(\nu)$ is a fourth-degree polynomial in ν:

$$\begin{aligned} R(\nu)={}&\nu^4-4s_1\nu^3+(6s_1^2-8s_2\pm48\sqrt{s_4})\nu^2\\ &-(4s_1^3-16s_1s_2+64s_3\pm32s_1\sqrt{s_4})\nu+(-s_1^2+4s_2\pm8\sqrt{s_4})^2. \end{aligned} \tag{3.15}$$

The root $\sqrt{s_4}$ in these formulae is considered to be $\sqrt{s_4}=\lambda_1\lambda_2\lambda_3\lambda_4$. We shall call the polynomials $R(\nu)$ the 'resolvents' of polynomial $P(\lambda)$, since their zeros are related to the zeros of $P(\lambda)$ by simple symmetric formulae: the zeros

$$\begin{aligned} &\nu_1=(\lambda_1+\lambda_2+\lambda_3-\lambda_4)^2 && \nu_2=(\lambda_1+\lambda_2-\lambda_3+\lambda_4)^2 \\ &\nu_3=(\lambda_1-\lambda_2+\lambda_3+\lambda_4)^2 && \nu_4=(-\lambda_1+\lambda_2+\lambda_3+\lambda_4)^2 \end{aligned} \tag{3.16}$$

correspond to the upper signs in (3.15), and the zeros

$$\begin{aligned} &\nu_1=(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^2 && \nu_2=(\lambda_1+\lambda_2-\lambda_3-\lambda_4)^2 \\ &\nu_3=(\lambda_1-\lambda_2+\lambda_3-\lambda_4)^2 && \nu_4=(-\lambda_1+\lambda_2+\lambda_3-\lambda_4)^2 \end{aligned} \tag{3.17}$$

correspond to the lower signs. This can be proved by a simple check of the Viete formulae. The passage mentioned above to the sixth-degree polynomial $P(\lambda)$ can be accomplished by removing one of the zeros λ_i.

As follows from (3.12) and first formula (3.13), the variable μ depends only on the phase

$$W=x+2s_1t \qquad \frac{d\mu}{dW}=\pm 4i\sqrt{P(\mu^{1/2})}=\pm 4if(\mu^{1/2}). \tag{3.18}$$

Differentiation of $P(\mu^{1/2})=f^2(\mu^{1/2})$ with respect to ν gives

$$\frac{dP(\mu^{1/2})}{d\mu}\frac{d\mu}{d\nu}=2f(\mu^{1/2})\left(2\mu\frac{d\mu}{d\nu}+\frac{\mu}{2}+\frac{\nu-s_1}{2}\frac{d\mu}{d\nu}\right)$$

which leads to the expression for the derivative

$$\frac{d\nu}{d\mu}=\frac{i\sqrt{-R(\nu)}}{4f(\mu^{1/2})}. \tag{3.19}$$

Multiplying (3.18) by (3.19), we find the equation for ν:

$$\frac{d\nu}{dW}=\sqrt{-R(\nu)}. \tag{3.20}$$

This equation can be easily resolved by means of elliptic functions. If ν is known, then $u(x, t)$ can be obtained from (3.10). With the help of (3.13) we get

$$\frac{\partial u}{\partial t}=16i\sqrt{s_4}\,u+2s_1\frac{\partial u}{\partial x}$$

so that

$$u(x,t)=\exp(16i\sqrt{s_4}\,t)\hat{u}(W) \tag{3.21}$$

where $\hat{u}(W)$ should satisfy the equation

$$\frac{d\hat{u}}{dW}=4i(-\tfrac{1}{2}s_1+\tfrac{1}{2}\nu+\mu)\hat{u}.$$

Substitution of (3.14) and (3.20) results in the equation

$$\frac{d\log\hat{u}}{dW}=\frac{1}{2}\frac{d\log\nu}{dW}+4i[-\tfrac{1}{4}s_1+\tfrac{3}{8}\nu+\tfrac{1}{8}(\nu_1\nu_2\nu_3\nu_4)^{1/2}(1/\nu)] \tag{3.22}$$

where the identity, following from (3.15),

$$(4s_2 \pm 8s_4^{1/2} - s_1^2)^2 = \nu_1\nu_2\nu_3\nu_4$$

was also taken into account.

It is clear that the zeros λ_i should be numbers such that ν oscillates between two positive values. The polynomial $R(\nu)$ has four zeros ν_i, which are given by (3.16) or (3.17) depending on the choice of sign in (3.15). If only two ν_i are real and positive, then let us enumerate λ_i so that these ν_i are ν_1, ν_2 and $\nu_1 \geqslant \nu_2$. If all the ν_i are real and positive, then let us enumerate λ_i so that $\nu_1 \geqslant \nu_2 \geqslant \nu_3 \geqslant \nu_4$. Thus, as is clear from (3.20), the variable ν can oscillate in the intervals $\nu_1 \geqslant \nu \geqslant \nu_2$ or $\nu_3 \geqslant \nu \geqslant \nu_4$, where $R(\nu) \leqslant 0$.

Let us list the λ_i, $i = 1, 2, 3, 4$, corresponding to the periodic solutions.

(i) The zeros λ_i consist of two complex conjugate pairs:

$$\lambda_1 = \alpha + \mathrm{i}\beta \qquad \lambda_2 = \alpha - \mathrm{i}\beta \qquad \lambda_3 = \gamma - \mathrm{i}\delta \qquad \lambda_4 = \gamma + \mathrm{i}\delta. \tag{3.23}$$

Then (3.17) yields

$$\begin{aligned} \nu_1 &= 4(\alpha+\gamma)^2 & \nu_2 &= 4(\alpha-\gamma)^2 \\ \nu_3 &= -4(\beta-\gamma)^2 & \nu_4 &= -4(\beta+\gamma)^2 \end{aligned} \tag{3.24}$$

and (3.16) results in the complex values of ν_i.

(ii) If

$$\lambda_1 = \alpha + \mathrm{i}\beta \qquad \lambda_2 = \alpha - \mathrm{i}\beta \qquad \lambda_3 = \gamma - \mathrm{i}\delta \qquad \lambda_4 = -\gamma - \mathrm{i}\delta \tag{3.25}$$

then (3.16) yields (3.24), and (3.17) becomes inapplicable. Both these cases correspond to the same solution, for which the variable ν oscillates in the interval $\nu_1 \geqslant \nu \geqslant \nu_2$.

(iii) All four λ_i are real and

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4.$$

Both (3.16) and (3.17) yield the real and positive ν_i corresponding to different periodic solutions for which the variable ν oscillates in the intervals $\nu_1 \geqslant \nu \geqslant \nu_2$ or $\nu_3 \geqslant \nu \geqslant \nu_4$.

(iv) If two λ_i are complex conjugate and two others are real

$$\lambda_1 = \alpha + \mathrm{i}\beta \qquad \lambda_2 = \alpha - \mathrm{i}\beta \qquad \lambda_3 = \gamma \qquad \lambda_4 = \delta \tag{3.26}$$

then (3.16) yields

$$\begin{aligned} \nu_1 &= (2\alpha+\gamma-\delta)^2 & \nu_2 &= (2\alpha-\gamma+\delta)^2 \\ \nu_3 &= (\gamma+\delta+2\mathrm{i}\beta)^2 & \nu_4 &= (\gamma+\delta-2\mathrm{i}\beta)^2 \end{aligned} \tag{3.27}$$

and (3.17) leads to the same values of ν_i with different signs before δ.

Now we shall turn to finding the periodic solutions. Let us discuss at first the case when the variable ν oscillates in the interval $\nu_1 \geqslant \nu \geqslant \nu_2$ and ν_3, ν_4 are also real. We shall choose initial conditions so that $\nu = \nu_1$ at $W = 0$. Then (3.20) leads to the solution (Gradshtein and Ryzhik 1980)

$$\frac{(\nu_2-\nu_4)(\nu_1-\nu)}{(\nu_1-\nu_2)(\nu-\nu_4)} = \mathrm{sn}^2\left(\sqrt{(\nu_1-\nu_3)(\nu_2-\nu_4)}\,\frac{W}{2},\, k\right) \tag{3.28}$$

where the elliptic function modulus is given by

$$k^2 = \frac{(\nu_1-\nu_2)(\nu_3-\nu_4)}{(\nu_1-\nu_3)(\nu_2-\nu_4)}. \tag{3.29}$$

The following calculations take a more convenient form in terms of the Weierstrass elliptic functions. Therefore we introduce the zeros of the Weierstrass cubic by means of the expressions

$$\begin{aligned} e_1 &= \tfrac{1}{12}[2(\nu_1-\nu_3)(\nu_2-\nu_4)-(\nu_1-\nu_2)(\nu_3-\nu_4)] \\ e_2 &= \tfrac{1}{12}[2(\nu_1-\nu_2)(\nu_3-\nu_4)-(\nu_1-\nu_3)(\nu_2-\nu_4)] \\ e_3 &= -\tfrac{1}{12}[(\nu_1-\nu_2)(\nu_3-\nu_4)+(\nu_1-\nu_3)(\nu_2-\nu_4)]. \end{aligned} \tag{3.30}$$

Then, taking into account the formula

$$\mathrm{sn}^2\left(\sqrt{(\nu_1-\nu_3)(\nu_2-\nu_4)}\,\frac{W}{2},\,k\right)=\frac{e_1-e_3}{\wp(W)-e_3}$$

we get from (3.28) the expression for ν:

$$\nu=\nu_1\frac{\wp(W)-\wp(\rho)}{\wp(W)-\wp(\kappa)}. \tag{3.31}$$

The parameters $\varkappa$ and ρ are defined by the equations

$$\wp(\varkappa)=e_3-\tfrac{1}{4}(\nu_1-\nu_2)(\nu_1-\nu_3) \qquad \wp(\rho)=e_3-\tfrac{1}{4}(\nu_4/\nu_1)(\nu_1-\nu_2)(\nu_1-\nu_3) \tag{3.32}$$

and the corresponding values of the derivatives are given by

$$\begin{aligned} \wp'(\varkappa) &= \frac{1}{4\mathrm{i}}(\nu_1-\nu_2)(\nu_1-\nu_3)(\nu_1-\nu_4) \\ \wp'(\rho) &= \frac{(\nu_1\nu_2\nu_3\nu_4)^{1/2}}{4\mathrm{i}\nu_1^2}(\nu_1-\nu_2)(\nu_1-\nu_3)(\nu_1-\nu_4). \end{aligned} \tag{3.33}$$

Now, after substitution of (3.31) into (3.22), one can integrate the equation by means of (2.26). After simple calculations, we get the expression for the periodic solution of the DNLS equation:

$$\begin{aligned} u(x,t) &= \exp\left[\mathrm{i}\left(-s_1+\frac{3\nu_1}{2}+\frac{(\nu_1\nu_2\nu_3\nu_4)^{1/2}}{2\nu_1}\right)W+(3\zeta(\varkappa)-\zeta(\rho))W+16\mathrm{i}\sqrt{s_4}\,t\right] \\ &\quad\times\sqrt{\nu_1}\,\frac{\sigma(\varkappa)\sigma(W+\rho)\sigma(W-\varkappa)}{\sigma(\rho)\sigma^2(W+\varkappa)} \qquad \nu_1\geqslant\nu\geqslant\nu_2. \end{aligned} \tag{3.34}$$

The case when ν oscillates in the interval $\nu_3\geqslant\nu\geqslant\nu_4$ can be considered in the same way. Initial conditions are chosen so that $\nu=\nu_4$ at $W=0$. For ν we get the expression

$$\nu=\nu_4\frac{\wp(W)-\wp(\rho)}{\wp(W)-\wp(\varkappa)} \tag{3.35}$$

where $\varkappa$ and ρ are now defined by

$$\wp(\varkappa)=e_3-\tfrac{1}{4}(\nu_2-\nu_4)(\nu_3-\nu_4) \qquad \wp(\rho)=e_3-\tfrac{1}{4}(\nu_1/\nu_4)(\nu_2-\nu_4)(\nu_3-\nu_4). \tag{3.36}$$

The corresponding periodic solution of the DNLS equation takes the form

$$\begin{aligned} u(x,t) &= \exp\left[\mathrm{i}\left(-s_1+\frac{3\nu_4}{2}+\frac{(\nu_1\nu_2\nu_3\nu_4)^{1/2}}{2\nu_4}\right)W+(3\zeta(\varkappa)-\zeta(\rho))W+16\mathrm{i}\sqrt{s_4}\,t\right] \\ &\quad\times\sqrt{\nu_4}\,\frac{\sigma(\varkappa)\sigma(W+\rho)\sigma(W-\varkappa)}{\sigma(\rho)\sigma^2(W+\varkappa)} \qquad \nu_3\geqslant\nu\geqslant\nu_4. \end{aligned} \tag{3.37}$$

The solutions (3.34) and (3.37) correspond to the λ_i listed above as (i)-(iii). Case (iv) cannot be described by (3.34), since the modulus (3.29) is complex in this case. We shall obtain here only the expression for ν. For the case of the resolvent's zeros (3.27), equation (3.20) takes the form

$$\frac{\mathrm{d}\nu}{\mathrm{d}W}=\{(\nu_1-\nu)(\nu-\nu_2)[(\nu-m)^2+n^2]\}^{1/2} \tag{3.38}$$

where

$$m=(\gamma-\delta)^2+n^2 \qquad n=2\beta(\gamma-\delta). \tag{3.39}$$

The result of integrating (3.38) can be expressed through the Jacobi elliptic functions as follows (Gradshtein and Ryzhik 1980):

$$\nu=\frac{\nu_1 p_2\,\mathrm{sn}^2(\hat\vartheta,\hat k)+\nu_2 p_1(1-\mathrm{cn}(\hat\vartheta,\hat k))^2}{p_2\,\mathrm{sn}^2(\hat\vartheta,\hat k)+p_1(1-\mathrm{cn}(\hat\vartheta,\hat k))^2} \tag{3.40}$$

where

$$\begin{gathered} p_{1,2}^2=(m-\nu_{1,2})^2+n^2 \qquad \hat\vartheta=(p_1p_2)^{1/2}W \\ W=x+2(2\alpha^2-2\beta^2+\gamma^2+\delta^2)t \\ \hat k^2=[(\nu_1-\nu_2)^2-(p_1-p_2)^2]/(4p_1p_2). \end{gathered} \tag{3.41}$$

Now let us discuss some limiting cases for these periodic solutions.

3.3. *Limiting cases*

We shall first consider the soliton limit of (3.34) when $\nu_2=\nu_3$, i.e., $k=1$ and

$$e_1=e_2=a=\tfrac{1}{12}(\nu_1-\nu_2)(\nu_2-\nu_4) \qquad e_3=-2a=-\tfrac{1}{6}(\nu_1-\nu_2)(\nu_2-\nu_4). \tag{3.42}$$

By means of well known limiting expressions for the Weierstrass functions (Erdelyi *et al* 1955), we obtain from (3.34)

$$\begin{aligned} u(x,t)=-\exp\{&\mathrm{i}[-s_1+\tfrac{3}{2}\nu_1+\tfrac{1}{2}\nu_2(\nu_4/\nu_1)^{1/2}]W \\ &+[3\sqrt{3a}\coth(\sqrt{3a}\,\varkappa)-\sqrt{3a}\coth(\sqrt{3a}\,\rho)]W+16\mathrm{i}\sqrt{s_4}\,t\} \\ &\times\frac{\sqrt{\nu_1}\sinh[\sqrt{3a}(W+\rho)]\sinh[\sqrt{3a}(W-\varkappa)]}{\sinh(\sqrt{3a}\,\rho)\sinh^2[\sqrt{3a}(W+\varkappa)]}. \end{aligned} \tag{3.43}$$

Equations (3.32) give in this limit

$$\begin{aligned} &\sqrt{3a}\coth(\sqrt{3a}\,\varkappa)=-\tfrac{1}{2}\mathrm{i}(\nu_1-\nu_2) \\ &\sqrt{3a}\coth(\sqrt{3a}\,\rho)=-\tfrac{1}{2}(\nu_4/\nu_1)^{1/2}(\nu_1-\nu_2). \end{aligned} \tag{3.44}$$

Let us denote

$$2\vartheta=\sqrt{3a}\,W \qquad \cos^2\frac{\Gamma}{2}=\frac{\nu_2-\nu_4}{\nu_1-\nu_4} \tag{3.45}$$

so that

$$\begin{aligned} &\sinh(\sqrt{3a}\,\varkappa)=\mathrm{i}\cos(\Gamma/2) \qquad \cosh(\sqrt{3a}\,\varkappa)=\sin(\Gamma/2) \\ &\sinh(\sqrt{3a}\,\rho)=\mathrm{i}\sqrt{\nu_1}\cos(\tfrac{1}{2}\Gamma)(\nu_1\cos^2(\tfrac{1}{2}\Gamma)+\nu_4\sin^2(\tfrac{1}{2}\Gamma))^{-1/2} \\ &\cosh(\sqrt{3a}\,\rho)=\sqrt{\nu_4}\sin(\tfrac{1}{2}\Gamma)(\nu_1\cos^2(\tfrac{1}{2}\Gamma)+\nu_4\sin^2(\tfrac{1}{2}\Gamma))^{-1/2}. \end{aligned} \tag{3.46}$$

Simple transformations of (3.43) with the use of (3.44)-(3.46) result in the soliton-like solution

$$u(x,t)=\tfrac{1}{2}\exp[\mathrm{i}(-s_1+\tfrac{3}{2}\nu_2)W+16\mathrm{i}\sqrt{s_4}\,t-\tfrac{1}{2}\sqrt{-\nu_1\nu_4}\,W]$$
$$\times\frac{\cosh(2\vartheta+\mathrm{i}\Gamma/2)}{\cosh(2\vartheta-\mathrm{i}\Gamma/2)}\left(\sqrt{\nu_1}+\sqrt{\nu_2}+(\sqrt{\nu_1}-\sqrt{\nu_4})\frac{\cosh(2\vartheta+\mathrm{i}\Gamma/2)}{\cosh(2\vartheta-\mathrm{i}\Gamma/2)}\right). \tag{3.47}$$

This solution describes a soliton propagating on the constant background.

Let us discuss two particular cases of this solution. Consider the case

$$\lambda_1=\lambda_4=\alpha+\mathrm{i}\beta \qquad \lambda_2=\lambda_3=\alpha-\mathrm{i}\beta \tag{3.48}$$

so that

$$\nu_1=16\alpha^2 \qquad \nu_2=\nu_3=0 \qquad \nu_4=-16\beta^2 \qquad \cos^2(\tfrac{1}{2}\Gamma)=\beta^2/(\alpha^2+\beta^2).$$

The last formula prompts the parametrisation

$$\alpha=\Delta\sin(\tfrac{1}{2}\Gamma) \qquad \beta=\Delta\cos(\tfrac{1}{2}\Gamma). \tag{3.49}$$

Substitution of these expressions into (3.47) leads to the soliton

$$u(x,t)=4\Delta\sin\Gamma\frac{\exp(2\mathrm{i}\Phi)}{\exp(2\vartheta)+\exp(-2\vartheta+\mathrm{i}\Gamma)}\,\frac{\exp(4\vartheta)+\exp(-\mathrm{i}\Gamma)}{\exp(4\vartheta)+\exp(\mathrm{i}\Gamma)} \tag{3.50}$$

$$\Phi=2\Delta^2(\cos\Gamma)x-8\Delta^4(\cos2\Gamma)t \qquad \vartheta=2\Delta^2(\sin\Gamma)x-8\Delta^4(\sin2\Gamma)t \tag{3.51}$$

which differs from the soliton of Kaup and Newell (1978) only through a different notation.

Let now all the λ_i be real and equal to

$$\lambda_1=\tfrac{1}{2}(\alpha+\beta) \qquad \lambda_2=\lambda_3=\tfrac{1}{2}\beta \qquad \lambda_4=-\tfrac{1}{2}(\alpha-\beta) \tag{3.52}$$

so that

$$\nu_1=4\beta^2 \qquad \nu_2=\nu_3=\alpha^2 \qquad \nu_4=0 \qquad \cos^2(\tfrac{1}{2}\Gamma)=(\alpha^2/4\beta^2)$$

and substitution into (3.47) gives

$$u(x,t)=\alpha\,\mathrm{e}^{\mathrm{i}\Phi}\frac{\cosh(2\vartheta)\cosh(2\vartheta+\mathrm{i}\Gamma/2)}{\cosh^2(2\vartheta-\mathrm{i}\Gamma/2)} \tag{3.53}$$

where

$$\begin{aligned}\Phi&=(\alpha^2+\beta^2)x+[(\alpha^2+\beta^2)^2-4\beta^4]t\\ \vartheta&=\tfrac{1}{4}\alpha(4\beta^2-\alpha^2)^{1/2}[x+(\alpha^2+2\beta^2)t].\end{aligned} \tag{3.54}$$

This is the 'bright' soliton on the background, as one can see from the expression for squared modulus of field:

$$\nu=|u(x,t)|^2=4\alpha^2\beta^2[\alpha^2+(4\beta^2-\alpha^2)\tanh^2 2\vartheta]^{-1}. \tag{3.55}$$

In the same way, one can consider the soliton limit of periodic solution (3.37). The final result for the case (3.52) has the form

$$u(x,t)=\mathrm{i}\alpha\,\mathrm{e}^{\mathrm{i}\Phi}\frac{\sinh(2\vartheta)\cosh(2\vartheta-\mathrm{i}\Gamma/2)}{\cosh^2(2\vartheta+\mathrm{i}\Gamma/2)} \tag{3.56}$$

where $\sin^2(\Gamma/2) = \alpha^2/(4\beta^2)$ and Φ, ϑ are given by (3.54). The squared modulus of the field is given by

$$\nu = |u(x, t)|^2 = 4\alpha^2\beta^2[\alpha^2 + (4\beta^2 - \alpha^2)\coth^2 2\vartheta]^{-1}. \tag{3.57}$$

This is the 'dark' soliton on the constant background.

4. Conclusion

We see that the suggested integration method with the use of the resolvents of the polynomial $P(\lambda)$ is rather effective and can be applied to the different integrable equations. One can suppose that there exists a generalisation of the method which will permit one to obtain the effective formulae for multi-phase (particularly, two-phase) solutions of integrable equations.

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